Kuwait University — Math. 102 — May 27, 1999

Dept. of Math. & Comp. Sci. — Final Exam — Duration: 2 hours

Calculators, Mobile phones and Pagers ARE NOT ALLOWED

Each question/part is worth 4 points.

1. Given $f(x) = xe^{x^2} + 2\sin^{-1} x + \sinh x^3 + 1$, $-1 \le x \le 1$. Show that f^{-1} exists and find the equation of the tangent line to the graph of f^{-1} at the point $P(1,0)$.

2. Find $\lim\limits_{x\to 0} \dfrac{x+\sin x}{\tan^{-1} x}$.

3. Evaluate $\displaystyle\int_1^{\infty} \frac{1}{x(x^2+1)}\,dx$ if it is convergent.

4. Evaluate the following integrals

   (a) $\displaystyle\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

   (b) $\displaystyle\int \frac{2x+3}{x^2-4x+8}\,dx$

   (c) $\displaystyle\int (\cosh x)^{-2}\sinh^3 x\,dx$

   (d) $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

5. Find the length of the curve whose parametric equations are

$$x = \frac{1}{2}e^{2t}, \quad y = \frac{1}{3}e^{3t}, \quad \ln\sqrt{2} \le t \le \ln\sqrt{8}.$$

6. Find the area of the region that is inside the graphs of both equations $r = \sqrt{3}\sin\theta$, and $r = \cos\theta$.

7. Find the equation of the plane parallel to the vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and containing the points $P(1,-2,2)$ and $Q(2,-1,3)$.

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Solution Key

1. $f'(x) = (1+2x^2)e^{x^2} + 2\frac{1}{\sqrt{1-x^2}} + 3x^2\cosh x^3 > 0 \Rightarrow f^{-1}$ exists.

   $f(0) = 1$, so P is on the graph of f^{-1}.

   $\frac{df^{-1}}{dx}(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{1+2} = \frac{1}{3}.$

   Equation: $3y - x + 1 = 0.$

2. $\frac{0}{0}$ type at 0.

   $\frac{(x+\sin x)'}{(\tan^{-1} x)'} = \frac{1+\cos x}{\frac{1}{1+x^2}} = (1+x^2)(1+\cos x).$

   $\lim_{x\to 0}(1+x^2)(1+\cos x) = 2.$

   By L'hospital's Rule $\lim_{x\to 0}\frac{x+\sin x}{\tan^{-1} x} = 2.$

3. $\int_1^c \frac{1}{x(x^2+1)}dx = \int_1^c \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx = \ln x - \frac{1}{2}\ln(1+x^2)\Big|_1^c$

   $= \ln \frac{x}{\sqrt{1+x^2}}\Big|_1^c = \ln\frac{c}{\sqrt{1+c^2}} - \ln\frac{1}{\sqrt{2}}.$

   $\lim_{c\to\infty}\left(\ln\frac{c}{\sqrt{1+c^2}} - \ln\frac{1}{\sqrt{2}}\right) = \ln\sqrt{2} \Rightarrow \int_1^\infty \frac{1}{x(x^2+1)}dx = \ln\sqrt{2}.$

4. (a) Let $u = \sin^{-1} x$, $dv = \frac{x}{\sqrt{1-x^2}} dx$ $du = \frac{dx}{\sqrt{1-x^2}}$ $v = -\sqrt{1-x^2}$

$$\begin{aligned}\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx &= \int u dv = uv - \int v du \\ &= -\sqrt{1-x^2} \sin^{-1} x + \int dx \\ &= -\sqrt{1-x^2} \sin^{-1} x + x + C.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2x+3}{(x-2)^2+4} dx &= \int \frac{2x+3}{(x-2)^2+4} dx \\ &= \int \frac{2(u+2)+3}{u^2+4} du \quad [u = x-2] \\ &= \int \frac{2u+7}{u^2+3} du = \int \frac{2u}{u^2+4} du + \int \frac{7}{u^2+4} du \\ &= \ln\left(x^2-4x+8\right) + \frac{7}{2} \tan^{-1} \frac{x-2}{2} + C.\end{aligned}$$

(c)

$$\begin{aligned}\int (\cosh x)^{-3} \sinh^3 x dx &= \int \left[(\cosh x)^{-3} \sinh^2 x\right] \sinh x dx \\ &= \int \left[(\cosh x)^{-3} \left(\cosh^2 x - 1\right)\right] \sinh x dx \\ &= \int u^{-3} \left(u^2 - 1\right) du \quad [u = \cosh x] \\ &= \ln u + \frac{1}{2} u^{-2} + C \\ &= \ln \cosh x + \frac{1}{2} (\cosh x)^{-2} + C\end{aligned}$$

(d)

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}} dx &= \int \frac{\tan^3 \theta}{\sec \theta} \sec^2 \theta d\theta \quad [x = \tan \theta] \\ &= \int \left(\tan^2 \theta\right) \tan \theta \sec \theta d\theta \\ &= \int \left(\sec^2 \theta - 1\right) \tan \theta \sec \theta d\theta = \frac{1}{3} \sec^3 \theta - \sec \theta + C \\ &= \frac{1}{3} \left(1+x^2\right)^{\frac{3}{2}} - \sqrt{1+x^2} + C.\end{aligned}$$